FINITE-DIFFERENCE SOLUTION OF CONJUGATE HEAT-CONDUCTION PROBLEMS

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The nonlinear conjugate heat-transfer problem is solved by the method of finite differences. The convergence of the resulting finite-difference scheme is analyzed.

Conjugate heat-transfer problems are receiving ever-increasing attention. The problems of liquid or gas flow past a plate and of fluid flow in a pipe have very broad applications in engineering, and it is not surprising that many attempts have been undertaken to find their solutions. An exhaustive bibliography on the topic may be found in [1, 2]. The indicated processes are described by a system of partial differential equations, which are of a variety of types. The problems turn out to be rather complex, and so mainly linear equations are discussed in the cited papers. In the present study we pose the problem of heat propagation in bodies with different temperature-dependent thermal conductivities. The problem is solved by the method of finite differences. We construct a finite-difference scheme and analyze its properties.

Let us consider the problem

$$\frac{\partial U_1}{\partial x}=\frac{a^2(\rho)}{\rho}\frac{\partial}{\partial\rho}\left[\rho\lambda_1(U_1)\frac{\partial U_1}{\partial\rho}\right]+f_1(\rho, x),\ 0<\rho<R; \tag{1}$$

$$U_1(\rho, 0)=\alpha(\rho); \tag{2}$$

$$\frac{\partial U_1(0, x)}{\partial\rho}=0; \tag{3}$$

$$\frac{\partial}{\partial x}\left[\lambda_2(U_2)\frac{\partial U_2}{\partial x}\right]+\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\lambda_2(U_2)\frac{\partial U_2}{\partial\rho}\right]=f_2(\rho, x),\ R<\rho<R_1; \tag{4}$$

$$U_2(\rho, 0)=\beta(\rho); \tag{5}$$

$$U_2(\rho, l)=\gamma(\rho); \tag{6}$$

$$U_1(R-0, x)=U_2(R+0, x); \tag{7}$$

$$\rho\lambda_1(U_1)\frac{\partial U_1}{\partial\rho}\Big|_{\rho=R-0}=\rho\lambda_2(U_2)\frac{\partial U_2}{\partial\rho}\Big|_{\rho=R+0}; \tag{8}$$

$$U_2(R_1, x)=\delta(x). \tag{9}$$

§1. In the cylinder $0 \le x \le l$, $0 \le \rho \le R$ we construct a grid of circles $\rho_i = ih$, where $R = Nh$, $R_1 = N_1h$, and the straight lines $x_k = k\tau$, $k = 0, 1, \ldots$, i.e., the lines $x_k = k\tau$ intersect the inner wall $\rho = R$ only at nodes of the grid.

We use the following approximation of the derivatives:

$$\frac{\partial U(\rho_i, x)}{\partial x}=\frac{U(\rho_i, x_{k-1})-U(\rho_i, x_k)}{\tau}+O(\tau); \tag{10}$$

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$$\frac{\partial}{\partial \rho}\left[\rho\lambda(\rho)\frac{\partial U}{\partial \rho}\right]=\frac{1}{h^2}\left[\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k}(U_{i+1,k+1}-U_{i,k+1})+\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k}(U_{i,k+1}-U_{i-1,k+1})\right]+O(h^2), \tag{11}$$

$$\frac{\partial}{\partial x}\left[\lambda(U)\frac{\partial U}{\partial x}\right]=\frac{1}{\tau^2}\left[\lambda_{i,k+\frac{1}{2}}(U_{i+1,k+1}-U_{i+1,k})-\lambda_{i,k-\frac{1}{2}}(U_{i+1,k}-U_{i+1,k-1})\right]+O(\tau^2). \tag{12}$$

At points of the common boundary $\rho = R$ we approximate condition (8) with regard for condition (7):

$$\rho\lambda_1(U_1)\frac{\partial U_1}{\partial \rho}\bigg|_{\substack{x=x_k\\ \rho=R-0}}=\rho_{N-\frac{1}{2}}\lambda_{N-\frac{1}{2},k-1}\frac{U_{N,k}-U_{N-k,k}}{h}+O(h), \tag{13}$$

$$\rho\lambda_2(U_2)\frac{\partial U_2}{\partial \rho}\bigg|_{\substack{x=x_k\\ \rho=R+0}}=\rho_{N+\frac{1}{2}}\lambda_{N+\frac{1}{2},k}\frac{U_{N+1,k}-U_{N,k}}{h}+O(h). \tag{14}$$

Inasmuch as heat transfer takes place in the ρ and x directions and does not depend on the polar angle φ, it is sufficient to carry out the analysis in one cross section $\varphi = \text{const}$, i.e., in the rectangle $D: 0 \le \rho \le R_1$, $0 \le x \le l$ with a cut along the line of discontinuity $\rho = R$.

The set of grid points belonging to D is denoted by D_h. We obtain the following difference problem approximating (1)-(9) with order $h + \tau$:

$$Y_{0,k}-Y_{1,k}=0, \tag{15}$$

$$\tau h^{-2}a^2_{i+\frac{1}{2}}\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k-1}Y_{i+\frac{1}{2},k}-[\rho_i+(a^2_{i+\frac{1}{2}}\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k-1}+$$
$$+a^2_{i-\frac{1}{2}}\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k-1})\tau h^{-2}]Y_{i,k}+\tau h^{-2}a_{i-\frac{1}{2}}\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k-1}Y_{i-1,k}=$$
$$=-\rho_i Y_{i,k-1}+\tau f^{(1)}_{i,k},\ i=1,\dots,N-1; \tag{16}$$

$$Y_{N+1,k}\rho_{N+\frac{1}{2}}\lambda_{N+\frac{1}{2},k}-(\rho_{N+\frac{1}{2}}\lambda_{N+\frac{1}{2},k}+\rho_{N-\frac{1}{2}}\lambda_{N-\frac{1}{2},k})Y_{N,k}+$$
$$+\rho_{N-\frac{1}{2}}\lambda_{N-\frac{1}{2},k}Y_{N-1,k}=0; \tag{17}$$

$$h^{-2}[\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k}Y_{i+1,k}-(\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k}+\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k})Y_{i,k}+$$
$$+\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k}Y_{i-1,k}]+\tau^{-2}[\lambda_{i,k+\frac{1}{2}}Y_{i,k+1}-(\lambda_{i,k+\frac{1}{2}}+\lambda_{i,k-\frac{1}{2}})Y_{i,k}+$$
$$+\lambda_{i,k-\frac{1}{2}}Y_{i,k-1}]=f^{(2)}_{i,k},\ i=N+1,\dots,N_1; \tag{18}$$

$$Y_{i,0}=\beta_i,\ Y_{i,K}=\gamma_i,\ i=N+1,\dots,N_1;\ Y_{N_1,k}=\delta_k. \tag{19}$$

§2. We investigate the convergence of the solution of the difference problem (15)-(19) to the solution of problem (1)-(9). We denote by $\varepsilon_{i,k}$ the difference between the exact solutions of problem (1)-(9) $U(\rho_i, x_k)$ and (15)-(19)

$$Y_{i,k}-U(\rho_i,x_k)\equiv\varepsilon_{i,k}. \tag{20}$$

We substitute (20) into (15)-(19) and expand the coefficients into series in powers of $\varepsilon_{i,k}$. Using (10)-(14), we obtain a system for the determination of

$$\varepsilon_{0,k}-\varepsilon_{1,k}=0;$$
$$\Lambda_1\varepsilon=\tau h^{-2}a^2_{i+\frac{1}{2}}\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k-1}\varepsilon_{i+1,k}+\tau h^{-2}a^2_{i-\frac{1}{2}}\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k-1}\varepsilon_{i-1,k}-$$
$$-[\rho_i+\tau h^{-2}(a^2_{i+\frac{1}{2}}\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k-1}+a^2_{i-\frac{1}{2}}\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k-1})]\varepsilon_{i,k}$$
$$=-\rho_i\varepsilon_{i,k-1}+O(h+\tau),\ i=1,\dots,N-1; \tag{21}$$
$$\Lambda_2\varepsilon=\rho_{N+\frac{1}{2}}\lambda_{N+\frac{1}{2},k}\varepsilon_{N+1,k}-(\rho_{N+\frac{1}{2}}\lambda_{N+\frac{1}{2},k}+\rho_{N-\frac{1}{2}}\lambda_{N-\frac{1}{2},k})\varepsilon_{N,k}+$$
$$+\rho_{N-\frac{1}{2}}\lambda_{N-\frac{1}{2},k}\varepsilon_{N-1,k}=O(h+\tau);$$

$$
\begin{gathered}
\Lambda_3\varepsilon = \tau^2 h^{-2}\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k}\varepsilon_{i+1,k} + \tau^2 h^{-2}\rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k}\varepsilon_{i-1,k} + \\
+ \lambda_{i,k+\frac{1}{2}}\varepsilon_{i,k+1} + \lambda_{i,k-\frac{1}{2}}\varepsilon_{i,k-1} - [\lambda_{i,k-\frac{1}{2}} + \lambda_{i,k+\frac{1}{2}} + \\
+ \tau^2 h^{-2}(\rho_{i+\frac{1}{2}}\lambda_{i+\frac{1}{2},k} - \rho_{i-\frac{1}{2}}\lambda_{i-\frac{1}{2},k})]\varepsilon_{i,k} = O(h+\tau), \\
i = N+1, \ldots, N_1; \\
\varepsilon_{i,0} = 0, \ i = 0, 1, \ldots, N, \ldots, N_1; \\
\varepsilon_{i,k} = 0, \ i = N+1, \ldots, N_1; \\
\varepsilon_{N_1,k} = 0.
\end{gathered}
\tag{21}
$$

The system of equations (21) can be written in the abbreviated form

$$
\begin{aligned}
&\varepsilon_{0,k} - \varepsilon_{1,k} = 0, \ \varepsilon_{i,k} = 0, \ i = N+1, \ldots, N_1; \\
&\Lambda_h\varepsilon = O(h+\tau), \ i = 1, \ldots, N_1 - 1; \\
&\varepsilon_{i,0} = \varepsilon_{N,k} = 0, \ i = 0, 1, \ldots, N_1;
\end{aligned}
\qquad
\Lambda_h \equiv \begin{cases} \Lambda_1, \ i = 1, \ldots, N-1 \\ \Lambda_2, \ i = N \\ \Lambda_3, \ i = N+1, \ldots, N_1 - 1. \end{cases}
\tag{22}
$$

Invoking the principle of frozen coefficients, we calculate the coefficients $\varepsilon_{i-1,k}$, $\varepsilon_{i+1,k}$, $\varepsilon_{i,k+1}$, $\varepsilon_{i,k-1}$, $\varepsilon_{i,k}$ at a certain fixed point. Then problem (22) becomes linear. Let us estimate its solution. To do so we verify that if

$$
\Lambda V \equiv AV_{i+1,k} + BV_{i-1,k} + CV_{i,k+1} + DV_{i,k-1} - (A+B+C+D)V_{ik} \geqslant 0 \tag{23}
$$

at interior points of the grid domain D_h, then the maximum (minimum) value of V is attained at the boundary.

We assume that the foregoing assertion is false and that the maximum is attained at interior points of the computing grid. We select from these points the one at which the coordinate ρ is largest. Let that point be (ρ_m, x_n). We write ΛV at this point in the form

$$
\Lambda V = A(V_{m+1,n} - V_{m,n}) + B(V_{m-1,n} - V_{m,n}) + C(V_{m,n+1} - V_{m,n}) + D(V_{m,n-1} - V_{m,n}).
$$

The first expression in parentheses is strictly less than zero, and all others are ≤ 0. Consequently, $\Lambda V < 0$, contradicting condition (23). From this verification we arrive at the maximum principle for the solutions of the equations $\Lambda V = 0$: All solutions of the equation $\Lambda V = 0$ attain their maximum and minimum values at the boundary of the domain.

Returning to the difference operators Λ_1, Λ_2, Λ_3, we see that Λ_3 has the same form as Λ. It follows from (21) that C = 0 and D = 0 for Λ_2 and that C = 0 for the operator Λ_1.

Hence, the maximum principle holds for the difference operator Λ_h. We show that the solution $\varepsilon_{i,k}$ of problem (22) satisfies the condition

$$
\max_{i,k}|\varepsilon_{i,k}| = O(h+\tau).
$$

For a system of the type

$$
\begin{gathered}
V_{i,k} = \beta_{i,k}, \\
(V_{i,k} - V_{0,k})/h = \alpha_k, \\
AV_{i+1,k} + BV_{i-1,k} + CV_{i,k+1} + DV_{i,k-1} - (A+B+C+D)V_{i,k} = \varphi_{i,k}
\end{gathered}
$$

with $i = N_1$, arbitrary k; arbitrary i, k = 0; or arbitrary i, k = K the following bound has been obtained [3]:

$$
\max_{i,k}|V_{i,k}| \leqslant M_1 \max_{i,k}|\varphi_{i,k}| + M_2 \max_{i,k}|\alpha_k| + M_3 \max_{i,k}|\beta_{i,k}|, \tag{24}
$$

where the coefficients M_1, M_2, M_3 are independent of h and τ. In our case $\alpha_k = 0$, $\beta_{i,k} = 0$, $\varphi_{i,k} = O(h+\tau)$. Using (24), we obtain

$$
\max_{\substack{0 \leqslant i \leqslant N_1 \\ 0 \leqslant k \leqslant K}} |\varepsilon_{i,k}| = O(h+\tau). \tag{25}
$$

Estimate (25) has been obtained on the assumption that the coefficients are calculated at a fixed point. Since (25) is valid at all points of the domain D_h, the principle of frozen coefficients implies convergence of the solution of the difference problem (15)-(19) to the solution of the differential problem (1)-(9) with order

$h + \tau$ in D_h. The order of the system of difference equations (15)-(19) is equal to the number of grid nodes, i.e., $N_1 \times K$.

The system cannot be partitioned into K independent subsystems as in the solution of the heat-conduction problem, due to the elliptical character of the differential equation in the annulus $R < \rho < R_1$. Methods for the solution of systems of the type (15)-(19) are described in adequate detail in [4]. We therefore omit any discussion of the algorithms for solution of the system of difference equations.

§3. In stating the conjugate problem (1)-(9) we have considered all the initial data for the problem to be known. It is only required to determine the temperature at the cylinder walls and in its interior. It is also important to consider conjugate problems in which, along with the temperature field, it is also required to determine the strength of a source in the interior or at the wall of the cylinder, the magnitude of a heat-flux jump, etc. We discuss one case of degeneracy of problem (1)-(9). We assume that at the cylinder wall heat transfer takes place only in the radial direction. This constraint will have a certain effect on the heat flux inside the cylinder. It is necessary to determine the temperature in the interior and at the wall of the cylinder as well as the heat-flux jump across the inner wall of the cylinder $\rho = R$. We state the problem

$$\begin{gathered}
\frac{\partial U_1}{\partial x} = \frac{a^2(\rho)}{\rho}\cdot\frac{\partial}{\partial\rho}\left[\rho\lambda_1(U_1)\frac{\partial U}{\partial\rho}\right] + f^{(1)}(\rho, x), \quad 0<\rho<R, \\
\frac{\partial U(0, x)}{\partial x} = 0; \quad U_1(\rho, 0) = \varphi(\rho); \\
f^2(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\lambda_2(U_2)\frac{\partial U_2}{\partial\rho}\right], \quad R<\rho<R_1, \\
\rho\lambda_1(U_1)\frac{\partial U_1}{\partial\rho}\bigg|_{\rho=R-0} = \rho\lambda_2(U_2)\frac{\partial U_2}{\partial\rho}\bigg|_{\rho=R+0} + \Psi(x), \\
U_1(R-0, x) = U_2(R+0, x), \quad U_2(R_1) = a = \text{const.}
\end{gathered} \tag{26}$$

We seek a continuous function $U(\rho, x)$ that is a solution of problem (26) and a function $\Psi(x)$, continuous in the domain $x \in [0, \infty]$, such that (26) has a solution. Demanding satisfaction of the matching condition at the boundary ρ, we obtain

$$U_1(R, x) = U_2(R) = \varphi(R).$$

We form a difference scheme by analogy with the procedure in §1:

$$\begin{gathered}
Y_{1,k} - Y_{0,k} = 0, \\
\tau h^{-2} a^2_{i+\frac{1}{2}} \rho_{i+\frac{1}{2}} \lambda_{i+\frac{1}{2},k-1} Y_{i+1,k} + \tau h^{-2} a^2_{i-\frac{1}{2}} \rho_{i-\frac{1}{2}} \lambda_{i-\frac{1}{2},k-1} Y_{i-1,k} - \\
- [\rho_i + \tau h^{-2}(a^2_{i+\frac{1}{2}} \rho_{i+\frac{1}{2}} \lambda_{i+\frac{1}{2},k-1} + a^2_{i-\frac{1}{2}} \rho_{i-\frac{1}{2}} \lambda_{i-\frac{1}{2},k-1})] Y_{i,k} = -\rho_i Y_{i,k-1} + \tau f^{(1)}_{ik}; \\
Y_{i,0} = \varphi_i, \quad i = 0, 1, \ldots, N_1; \\
\rho_{N+\frac{1}{2}} \lambda_{N+\frac{1}{2}} Y_{N+1,k} - (\rho_{N+\frac{1}{2}} \lambda_{N+\frac{1}{2},k-1} + \rho_{N-\frac{1}{2}} \lambda_{N-\frac{1}{2},k-1}) Y_{N,k} \\
+ \rho_{N-\frac{1}{2}} \lambda_{N-\frac{1}{2},k-1} Y_{N-1,k} = h^2 \Psi_k; \\
\rho_{i+\frac{1}{2}} \lambda_{i+\frac{1}{2},k} Y_{i+1,k} - (\rho_{i+\frac{1}{2}} \lambda_{i+\frac{1}{2},k} + \rho_{i-\frac{1}{2}} \lambda_{i-\frac{1}{2},k}) Y_{i,k} + \rho_{i-\frac{1}{2}} \lambda_{i-\frac{1}{2},k} Y_{i-1,k} = 0; \\
Y_{N,k} = a.
\end{gathered} \tag{27}$$

In the fourth equation of the system (27) neither the coefficients nor the unknown function depend on x or, hence, on k, and this index is written to unify the notation.

In the domain D_h we obtain a system of $N_1 \times K$ difference equations in $N_1 \times K$ unknowns:

$$\Psi_k, \quad k = 1, \ldots, K; \quad U_{i,k}, \; i = 0, 1, \ldots, N-1, \; k = 1, \ldots, K;$$

$$U_i, \; i = N+1, \ldots, N_1.$$

As an illustration we examine the problem

$$\frac{\partial U_1}{\partial x} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho U_1^2 \frac{\partial U_1}{\partial \rho}\right] + \frac{0.5(\rho - R)^2}{[1.5\rho^2 + 1.5x(\rho - R)^2]^{1/3}} - 2(1 + x) - \frac{R}{\rho}x, \quad 0 < \rho < R;$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho U_2 \frac{\partial U_2}{\partial \rho}\right] = 0, \quad R < \rho < 2R;$$

$$\frac{\partial U_1(0, x)}{\partial \rho} = 0, \quad U_1(\rho, 0) = \sqrt[3]{2R^2 \ln 2 + (1.5)^{2/3} R^{4/3}};$$

$$U_1(R - 0, x) = U_2(R + 0, x), \quad U_2(2R) = \sqrt{2R^2 \ln 2 + (1.5)^{2/3} R^{4/3}};$$

$$\rho U_1^2 \frac{\partial U_1}{\partial \rho}\bigg|_{\rho = R - 0} - \rho U_2 \frac{\partial U_2}{\partial \rho}\bigg|_{\rho = R + 0} = \Psi(x).$$

It is required to find $U(\rho, x)$ and $\Psi(x)$. This problem has the known exact solution

$$U_1 = \sqrt[3]{[1.5\rho^2 + 1.5x(\rho - R)^2]^2},$$

$$U_2 = \sqrt{2R^2 \ln(\rho/R) + (1.5)^{2/3} R^{4/3}}, \quad \Psi(x) = 0.$$

To find an approximate solution we use the difference scheme (27) with $h = 0.1$, $\tau = 0.04$, and $R = 1$. We have carried out the numerical computation on a BÉSM-4 digital computer. We give the values of $Y_{i,k}$ for $K = 400$: $Y_{0,k} = 2.8843$, $Y_{2,k} = 2.4897$, $Y_{4,k} = 2.0715$, $Y_{6,k} = 1.6368$, $Y_{8,k} = 1.2431$, $Y_{10,k} = 1.1442$, $Y_{12,k} = 1.2920$, $Y_{14,k} = 1.4083$, $Y_{16,k} = 1.5004$, $Y_{18,k} = 1.5776$, $Y_{20,k} = 1.6425$. We have also made a comparison of the $Y_{i,k}$ for $K = 400$ with the exact solution for $x = 16$. The error turns out to be not greater than 0.007.

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EFFECT OF A BIPERIODIC SYSTEM OF PLANE INCLUSIONS ON A PLANE STEADY TEMPERATURE FIELD

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Determining the complex potential of a plane temperature field perturbed by a biperiodic system of thin inclusions reduces to the solution of a singular integrodifferential equation.

1. Suppose that a plane steady temperature field is perturbed by some finite system of cuts (lines) Γ_n, $n = \overline{1, N}$. Each line may be taken to be, e.g., a foreign inclusion (or crack) of sufficiently large extension (relative to its width), the thermal conductivity k_n of which differs from the thermal conductivity k of the basic medium, taken to be the complex-variable plane $z = x + iy$. The set of all the lines Γ_n is denoted by $\Gamma = \Gamma_1 + \ldots + \Gamma_N$.

Consider the problem of finding the temperature field perturbed by inclusions, assuming that the temperature in a homogeneous body (in the absence of inclusions) is determined by a given harmonic function $T_0(x, y) = \operatorname{Re} F(z)$.

The complex potential of the perturbed temperature field $W(t) = T + i\psi$, where ψ is the current function associated with the temperature T, will be found as the sum of a given function F(z) and a Cauchy-type integral of unknown density taken along the curve Γ

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